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HEAT CONDUCTION AND HEAT EXCHANGE IN TECHNOLOGICAL PROCESSES

FUNCTIONAL IDENTIFICATION OF THE NONLINEAR THERMAL-CONDUCTIVITY COEFFICIENT BY GRADIENT METHODS. I. CONJUGATE OPERATORS

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Consideration is given to the gradient methods of solution of the inverse heat-conduction problem on determination of the nonlinear coefficient λ(T) without its preliminary finite-dimensional approximation.

Introduction. Gradient methods of numerical solution of inverse heat-conduction problems have been developed in many works, mainly in [1–3]. In particular, the problem of identification of the nonlinear thermal-conductivity coefficient $\lambda(T)$ has been considered in [3–6]. In [1–3, 7, 8], gradient methods have been used for restoration and evaluation of the power of heat sources.

One problem frequently arising when gradient methods are used is numerical realization of the values of conjugate (adjoint) operators. For example, in the case of identification of $\lambda(T)$, the operator conjugate to the internal-superposition operator (other names [9]: the substitution operator, the weighted-shift operator, the operator of replacement of a variable, and the composite operator) is present in the scheme of the method of conjugate gradients. The well-known approach presented in [3] leads to a complex and difficult-to-control procedure of computation of the values of the operator conjugate to the internal-superposition operator. Therefore, a finite-dimensional approximation of the sought nonlinear coefficients by any system of basis functions has been used in [3] and in subsequent works, thus reducing inverse heat-conduction problems to a problem of restoration of a finite number of parameters. In this connection, such approaches to solution of inverse heat-conduction problems are frequently called parametric ones.

In the present work, we consider heat-conduction problems without preliminary approximation of the functions sought. Such an approach is conventionally called functional (or finite-dimensional) identification. Functional identification of the nonlinear thermal-conductivity coefficient by gradient methods is based on new representations of the operator conjugate to the internal-superposition operator; these representations enable one to obtain formulas of the values of a conjugate operator, which are convenient for numerical calculations. We note that similar representations were used earlier in the theory of controlled integro-differential and functional-differential systems [10–12].

The results of the work are presented in two papers. In the first paper, we describe the algorithm of functional identification of the coefficient $\lambda(T)$; the emphasis is on finding the gradient of the square of the residual functional for $\lambda(T)$ in the space $L_2[T^{(1)}, T^{(2)}]$ of functions summable with the square and in the Sobolev space $W_2[T^{(1)}, T^{(2)}]$ of absolutely continuous functions. In the second paper, we consider numerical realization of the algorithm and discuss results of the numerical experiment on restoration of $\lambda(T)$.

Scheme of Gradient Methods. Following mainly [1–5], we give formal schemes of the method of conjugate gradients and the method of quickest descent which are used for solution of inverse problems of mathematical physics. We note that rigorous substantiation of the gradient methods of solution of different classes of inverse heat-conduction problems can be found in [3].

Let the system of operator equations

$$F(T, \lambda) = 0 , \tag{1}$$

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$$y = L(T)\,, \tag{2}$$

where T and λ are the sought quantities ($T \in X$, $\lambda \in \Lambda$, X and Λ are the Hilbert spaces), be prescribed. In the general case, the nonlinear equation (1) represents the primal problem for the variable T. The inverse problem is in determining the input λ data for the primal problem from the output y data, which may be considered as a result of certain measurements of the variable T. The measurement method is modeled by the continuous operator L: $X \to Y$ acting from the Hilbert space X into the Hilbert space Y. In applications, the operator L is linear, as a rule, which will be assumed in what follows.

The primal problem (1) satisfies the Hadamard correctness conditions and consequently is solvable for T ($T = \varphi(\lambda)$). Based on this fact and on (2), we have the following equation for λ:

$$L(\varphi(\lambda)) \equiv A(\lambda) = y\,. \tag{3}$$

Next we assume that Eq. (3) is uniquely solvable for $\lambda \in G$ in a certain open domain $G \subseteq \Lambda$. Variational methods of solution of Eq. (3) are based on minimization of the square of the residual functional

$$J(\lambda) = \frac{1}{2}\,\|A(\lambda) - y\|_Y^2 \equiv \frac{1}{2}\,\langle A(\lambda) - y, A(\lambda) - y\rangle_Y\,, \tag{4}$$

where $\|\cdot\|_Y$ and $\langle\cdot,\cdot\rangle_Y$ are respectively the norm and the scalar product in the space Y.

Next we use the formula for computation of the gradient J'_{λ_n} of the functional (4) at the point λ_n, where n is the No. of iteration of the algorithm. Let the operator A be differentiable according to Frechet in the domain G. The theorem on the derivative of a composition of mappings yields (see also [2, 3])

$$J'_{\lambda_n} = \left(A'_{\lambda_n}\right)^* p_n\,, \quad p_n := y_n - y\,, \quad y_n := A(\lambda_n)\,. \tag{5}$$

Here A'_{λ_n} is the Frechet derivative of the operator A at the point λ_n and $(A'_{\lambda_n})^*$ is the operator conjugate to A'_{λ_n}. The action of the operator $(A'_{\lambda_n})^*$ on the element $w \in Y$ is determined by the equality

$$\langle A'_{\lambda_n} u, w\rangle_Y = \langle u, \left(A'_{\lambda_n}\right)^* w\rangle_\Lambda\,, \quad \forall u \in \Lambda\,. \tag{6}$$

For one variant of the conjugate-gradient method we write the following recurrence system [3, 13]:

$$\lambda_{n+1} = \lambda_n - \beta_n l_n\,, \quad l_n = J'_{\lambda_n} - \gamma_{n-1} l_{n-1}\,, \quad l_0 = J'_{\lambda_0}\,, \tag{7}$$

where λ_0 is the initial approximation of the sought quantity λ. The parameter γ_{n-1} is prescribed by one equality [13]:

$$\gamma_{n-1} = -\frac{\|J'_{\lambda_n}\|_\Lambda^2}{\|J'_{\lambda_{n-1}}\|_\Lambda^2}\,, \quad \gamma_{n-1} = -\frac{\langle J'_{\lambda_n}, J'_{\lambda_{n-1}} - J'_{\lambda_n}\rangle_\Lambda}{\|J'_{\lambda_{n-1}}\|_\Lambda^2}\,.$$

The descent parameter β_n is determined by the condition

$$J(\lambda_n - \beta_n l_n) = \min_{\beta>0} J(\lambda_n - \beta l_n)\,. \tag{8}$$

Setting $\gamma_{n-1} = 0$ in (7), we obtain the recurrence system from the method of quickest descent [3].

From the theoretical and experimental results [1–8, 14] it follows that the gradient algorithms applied to a number of inverse heat-conduction problems (including the inverse problems of restoration of thermophysical parameters) possess regularizing properties. The iteration No. may act as the regularization parameter, whereas the residual criterion should be used to halt the algorithm [2, 3].

For computation of the operator $(A'_{\lambda_n})^*$ it is convenient to use the initial system (1) and (2). Setting $\lambda = \lambda_n + u$ and $T = T_n + \nu$ (u and ν are the variations of the variables λ and T) and allowing for the equality $F(T_n, \lambda_n) = 0$, we represent Eq. (1) in the form

$$F'_{(T_n, \lambda_n)}(\nu, u) + \alpha(\nu, u) = 0 .$$

Here $F'(T_n, \lambda_n)$ is the Frechet derivative at the point (T_n, λ_n) of the mapping $F : X \times \Lambda \to Z$ (Z is the Hilbert space of F values, $\|\alpha(\nu, u)\|_Z = o\|(\nu, u)\|_{X\times\Lambda}$). In so doing, we assume that the mapping F is differentiable at the point (T_n, λ_n) according to Frechet. Since $F'(T_n, \lambda_n)$ is a linear operator, we may write it in the form $F'(T_n, \lambda_n) = (M_n, -K_n)$, where $M_n : X \to Z$ and $K_n : \Delta \to Z$ are the linear bounded operators representing partial derivatives with respect to T and λ respectively at the point (T_n, λ_n) of the operator F. By virtue of the theorem on the derivative of an explicitly prescribed mapping, we have $A'_{\lambda_n} = LM_n^{-1}K_n$. It immediately follows that the conjugate operator $(A'_{\lambda_n})^*$ allows the representation

$$\left(A'_{\lambda_n}\right)^* = K_n^* (M_n^*)^{-1} L^* . \tag{9}$$

Also, it may be stated that the value $z_n = (A'_{\lambda_n})^* p_n$ of the operator $(A'_{\lambda_n})^*$ on the element p_n is determined by the problem

$$M_n^* w - L^* p_n = 0 , \quad z_n = K_n^* w , \tag{10}$$

conjugate to the problem

$$M_n \nu - K_n u = 0 , \quad \tilde{y} = L\nu \tag{11}$$

for the variations ν and n of the variables T and λ at the point (T_n, λ_n).

In solving inverse problems by the conjugate-gradient method, certain difficulties may arise in computation of the parameter β_n according to condition (8). The following technique, which has shown a good performance in solving different classes of inverse heat-conduction problems for approximate computation of the quantity β_n, is proposed in [1–3]. Based on the approximation, $A(\lambda_n - \beta l_n) \approx A(\lambda_n) - \beta L\nu(l_n)$, $\beta \approx 0$, where $\nu(l_n) = M_n^{-1} K_n l_n$ is the solution of problem (11) for $u = l_n$, we obtain

$$J(\lambda_n - \beta l_n) \approx \frac{1}{2} \| p_n - \beta L\nu(l_n)\|_Y^2 = \frac{1}{2}\left(\|p_n\|_Y^2 - 2\beta \langle L\nu(l_n), p_n\rangle + \beta^2 \|L\nu(l_n)\|^2\right). \tag{12}$$

The condition of steadiness in β of the approximation (given in (12)) of the functional $J(\lambda_n - \beta l_n)$ yields the estimate for β_n:

$$\beta_n \approx \frac{\langle L\nu(l_n), p_n\rangle_Y}{\|L\nu(l_n)\|_Y^2} . \tag{13}$$

There can be other approaches for computation of the quantity β_n which are based on the methods of minimization of the single-variable function [13, 14].

Noteworthy is another property of invariance of formula (9) with respect to the replacement of the space X by the arbitrary space $\tilde{X}$ in the case of dense nesting of X into $\tilde{X}$. This means that M_n and L may be considered as unbounded operators in the space $\tilde{X}$, i.e., the domains of definition and the values on identical elements for the operators $M_n : X \to Z$ and $P_n = M_n O^{-1} : \tilde{X} \to Z$ (O is the operator of nesting [15] of X into $\tilde{X}$) coincide, which also holds for the pair of operators $L : X \to Y$ and $Q = LO^{-1} : \tilde{X} \to Y$. To prove the above property of invariance we note that $M_n = P_n O$ and $L = QO$; therefore, we have

$$\left(A'_{\lambda_n}\right)^* = (QO(P_n O)^{-1} K_n)^* = K_n^* (P_n^*)^{-1} (O^*)^{-1} O^* Q = K_n^* (P_n^*)^{-1} Q^* .$$

Consequently, we may use formula (9), assuming that M_n and L have been determined in the space $\tilde{X}$, and, in accordance with this, may compute the conjugate operators M_n^* and L^*. This circumstance is important for applications, since expressions for the conjugate operators M_n and L in such an interpretation are more convenient as a rule. For inverse heat-conduction problems we usually have $\tilde{X} = Z$, where Z is the appropriate space for the set of values of the operator F.

Gradient of the Residual Squared. We consider the problem of determination of the thermal-conductivity coefficient $\lambda(T)$ in the following formulation [3, 4]:

$$c\,(T)\,\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda\,(T)\,\frac{\partial T}{\partial x}\right),$$

$$T\,(x, 0) = T_0\,(x)\,,\quad T\,(0, t) = g_1\,(t)\,,\ T\,(b, t) = g_2\,(t)\,, \tag{14}$$

$$y\,(t) = T\,(x_*, t)\,.$$

Here $(x, t) \in \Omega = [0, b] \times [0, t_{\rm f}]$; $g_1(t)$, $g_2(t)$, and $y(t)$ are the prescribed functions. The point x_* lies within the segment $[0, b]$. Also, we assume the fulfillment of the conditions of sufficient smoothness of the coefficients c and λ and mating of the initial and boundary conditions [3].

The operator A corresponding to the system of equations (14) may be considered as the operator acting from either $L_2[T^{(1)}, T^{(2)}]$ into $L_2[0, t_{\rm f}]$ or from the Sobolev space $W_2^k[T^{(1)}, T^{(2)}]$ into $L_2[0, t_{\rm f}]$ ($T^{(1)} = \min\limits_{(x,t)\in\partial\Omega} T(x, t)$ and $T^{(2)} = \max\limits_{(x,t)\in\partial\Omega} T(x, t)$, $\partial\Omega$ is the boundary of the domain Ω). We consider both cases, setting $k = 1$ for simplification in the second case, which corresponds to an absolute continuity of the function $\lambda : [T^{(1)}, T^{(2)}] \to \mathbb{R}$ and is natural from the viewpoint of the physical formulation of the inverse problem.

We denote the solution of system (14) by T_n for $\lambda(T) = \lambda_n(T)$. An equation of the form $M_n\nu - K_n u = 0$ for system (14) represents the initial boundary-value problem [3]:

$$\frac{\partial}{\partial t}\,(c\,(T_n)\,\nu) - \frac{\partial^2}{\partial x^2}\,(\lambda_n\,(T_n)\,\nu) - \frac{\partial}{\partial x}\left(u\,(T_n)\,\frac{\partial T_n}{\partial x}\right) = 0\,, \tag{15}$$

$$\nu\,(x, 0) = 0\,,\quad \nu\,(0, t) = \nu\,(b, t) = 0\,.$$

Thus, the action of the operators K_{ni} ($i \in \{0, 1\}$) ($i = 0$ corresponds to the first case and $i = 1$ corresponds to the second case) is prescribed by the formula

$$(K_{ni}u)\,(x, t) = \frac{\partial}{\partial x}\left(u\,(T_n)\,\frac{\partial T_n}{\partial x}\right),$$

whereas the action of the operator $M_n : W_{2,0}^{2,1}(\Omega) \to L_2(\Omega)$ is prescribed by the formula

$$(M_n\nu)\,(x, t) = \frac{\partial}{\partial t}\,(c\,(T_n)\,\nu) - \frac{\partial^2}{\partial x^2}\,(\lambda_n\,(T_n)\,\nu)\,. \tag{16}$$

Here $W_{2,0}^{2,1}(\Omega)$ is the Sobolev space of the functions ν satisfying the conditions ν, ν_x, ν_t, $\nu_{xx} \in L_2(\Omega)$, $\nu(0, t) = \nu(b, t) = 0$, and $\nu(x, 0) = 0$.

Finally, the action of the operator $L : W_{2,0}^{2,1}(\Omega) \to L_2[0, t_{\rm f}]$ is determined by the formula

$$(L\nu)\,(t) = \nu\,(x_*, t) \equiv \int\limits_0^b \delta\,(x - x_*)\,\nu\,(x, t)\,dx\,,$$

where $\delta(x - x_*)$ is the Dirac function.

The above-noted invariance of formula (9) with respect to the replacement of the space X by the space Z (in this case $X = W_{2,0}^{2,1}(\Omega)$ and $Z = L_2(\Omega)$) yields that we may consider M_n and L as unbounded operators in the space $L_2(\Omega)$ with the domain of definition $W_{2,0}^{2,1}(\Omega)$. Hence, from (6) and (16), using the standard procedure based on the Lagrange formula, we obtain (see also [3]) a system of the form (10) for problem (14):

$$c\,(T_n)\,\frac{\partial w}{\partial t} + \lambda_n\,(T_n)\,\frac{\partial^2 w}{\partial x^2} - \delta\,(x - x_*)\,p_n\,(t) = 0\,,$$

$$w\,(x, t_f) = 0\,,\quad w\,(0, t) = w\,(b, t) = 0\,, \tag{17}$$

$$z_n = K_{ni}^* w\,.$$

Here $z_n := J'_{\lambda_n}$ is the gradient of the functional (4) at the point $\underline{\lambda_n}$. Thus, the operator L^* conjugate to L is prescribed as $(L^* p_n)(x, t) = \delta(x - x_*) p_n(t)$. As far as the operators K_{ni}^* ($i = 0, 1$) are concerned, in [3–5], consideration is given to the method of computation of K_{n0}^*, which is associated with the replacement of variables in a double integral. Since this method involves certain difficulties in numerical realization of the algorithm, in [3] and in other works, use is made of the finite-dimensional approximation

$$\lambda\,(T) \approx \sum_{i=1}^{m} c_i \varphi_i\,(T)\,,$$

where $\{\varphi_i \mid i = \overline{1, m}\}$ is the prescribed set of linearly independent basis functions and $(c_1, ..., c_m)$ is the vector of the coefficients sought. With such an approach, the initial infinite-dimensional problem is reduced to a finite-dimensional one, which is, in principle, consistent with numerical reduction of infinite-dimensional problems and may work for improvement of the numerical stability of solution of inverse heat-conduction problems. At the same time, this reduction has drawbacks due to the uncertainty in selection of both basis functions and the number of these functions, i.e., of the parameter m characterizing the degree of approximation of the sought function $\lambda(T)$. This problem may be solved to some extent with the use of qualitative a priori information on the solution sought [3, 5]. We propose another approach to construction of the conjugate operator K_{n0}^*, which enables us to obtain quite an acceptable numerical scheme of calculation of K_{n0}^* values. To do this will require the following statement:

Statement 1. For the arbitrary linear bounded operator $S : L_2(\Omega) \to L_2(T^{(1)}, T^{(2)})$, we have the representation

$$(Sw)\,(z) = \frac{d}{dz}\int_{\Omega} (S^*\,\chi)\,(z, x, \tau)\,w\,(x, \tau)\,dx d\tau\,,$$

where $\chi(z, s) = \begin{cases} 1 & \text{for } T^{(1)} \le s \le z\,, \\ 0 & \text{for } z \le s \le T^{(2)}\,, \end{cases}$ χ is the characteristic function of the set $\{(z, s) \mid T^{(1)} \le s \le z \le T^{(2)}\}$, $(S^*\chi)(z, x, \tau)$ is the value at the point (x, τ) of action of the conjugate operator $S^* : L_2(T^{(1)}, T^{(2)}) \to L_2(\Omega)$ on the function $\chi(z, \cdot) : [T^{(1)}, T^{(2)}] \to \{0, 1\}$.

The proof of this statement immediately follows from the chain of equalities

$$(Sw)\,(z) = \frac{d}{dz}\int_{T^{(1)}}^{z} (Sw)\,(\tau)\,d\tau = \frac{d}{dz}\int_{T^{(1)}}^{T^{(2)}} \chi\,(z, \tau)\,(Sw)\,(\tau)\,d\tau = \frac{d}{dz}\int_{\Omega} (S^*\,\chi)\,(z, x, \tau)\,w\,(x, \tau)\,dx d\tau\,.$$

Proofs of the theorems given below are formed based on Statement 1.

Theorem 1. The $(K_{n0}^* w)(z)$ values of the conjugate operator K_{n0}^* may be computed from any of the formulas

$$(K_{n0}^{*}w)(z) = -\frac{d}{dz}\int_0^b\int_0^{t_f}\chi(z, T_n(x, t))\frac{\partial T_n(x, t)}{\partial x}\frac{\partial w(x, t)}{\partial x}dtdx \equiv -\frac{d}{dz}\int_{\omega(z)}\frac{\partial T_n(x, t)}{\partial x}\frac{\partial w(x, t)}{\partial x}dxdt \equiv$$

$$\equiv -\frac{d}{dz}\int_{\Omega}R(z, x, t)\frac{\partial T_n(x, t)}{\partial x}\frac{\partial w(x, t)}{\partial x}dxdt, \tag{18}$$

where $\omega(z) = \left\{(x, t) \in \Omega \,\middle|\, T_n(x, t) \le z \le T^{(2)}\right\}$ and $R(z, x, t)$ is the characteristic function of the set $w(z)$; the function $\omega(x, t)$ satisfies the boundary conditions $w(0, t) = w(b, t) = 0$.

Proof. We note that the operator K_{n0} represents the point of two operators, i.e., $K_{n0} = K_d K_n$, where $K_d = \frac{\partial}{\partial x}$ is the differential operator and K_n is the generalized operator of internal superposition: $(K_n u)(x, t) = \frac{\partial T_n(x, t)}{\partial x}u(T_n(x, t))$. Since $K_{n0}^{*} = K_n^{*}K_d^{*}$, applying Statement 1 to K_n^{*} and taking into account that $(K_n^{*})^{*} = K_n$ and $(K_d^{*}w)(x, t) = -\frac{\partial w(x, t)}{\partial x}$, $w(0, t) = w(b, t) = 0$, we obtain Theorem 1.

We pass to the construction of the conjugate operator K_{n1}^{*}. The operator K_{n1}^{*} acts from the space $L_2(\Omega)$ into $W_2^1[T^{(1)}, T^{(2)}]$. We may select the scalar product on the set of functions from $W_2^1[T^{(1)}, T^{(2)}]$ by different methods, for example, setting

$$\langle f, g\rangle_{W_2^1} = \int_{T^{(1)}}^{T^{(2)}} f(s)\, g(s)\, ds + \int_{T^{(1)}}^{T^{(2)}}\frac{df}{ds}\frac{dg}{ds}ds, \tag{19}$$

or

$$\langle f, g\rangle_{W_2^1} = f(T^{(1)})\, g(T^{(1)}) + \int_{T^{(1)}}^{T^{(2)}}\frac{df}{ds}\frac{dg}{ds}ds, \tag{20}$$

or

$$\langle f, g\rangle_{W_2^1} = f(T^{(2)})\, g(T^{(2)}) + \int_{T^{(1)}}^{T^{(2)}}\frac{df}{ds}\frac{dg}{ds}ds. \tag{21}$$

The norms generated by (19)–(21) are equivalent. We dwell on formulas (20) and (21), since the expression for the operator K_{n1}^{*} has the simplest form in this case.

Theorem 2. For the scalar product (20), the $(K_{n1}^{*}w)(z)$ values of the conjugate operator K_{n1}^{*} may be computed according to the formula

$$(K_{n1}^{*}w)(z) = -\int_{\Omega}\frac{\partial T_n(x, t)}{\partial x}\frac{\partial w(x, t)}{\partial x}dxdt - \int_{T^{(1)}}^{z}\int_{\Omega} r(\tau, x, t)\frac{\partial T_n(x, t)}{\partial x}\frac{\partial w(x, t)}{\partial x}dxdtd\tau, \tag{22}$$

where $r(\tau, x, t)$ is the characteristic function of the set $\overline{\omega}(\tau) = \Omega\backslash\omega(\tau) = \left\{(x, t) \in \Omega \,\middle|\, T^{(1)} \le \tau \le T_n(x, t)\right\}$.

Proof. Let us consider the case of the scalar product (20). Just as in proving Theorem 1, we represent the operator K_{n1} in the form of the product $K_{n1} = K_d K_{na}$, where $K_d = \frac{\partial}{\partial x}$, $K_{na} : W_2^1[T^{(1)}, T^{(2)}] \to L_2(\Omega)$, and $(K_{na}u)(x, t) = u(T_n(x, t))\frac{\partial T_n(x, t)}{\partial x}$. Since $K_{n1}^* = K_{na}^* K_d^*$, to compute the operator K_{n1}^* we must only obtain expressions for K_{na}^*. For the description of K_{na}^* we note that K_{na} may be represented as the loaded integro-differential operator

$$(K_{na}u)(x, t) = T_n(x, t)\left(u(T^{(1)}) + \int_{T^{(1)}}^{T_n(x, t)} \frac{du(\tau)}{d\tau}\, d\tau\right) \equiv T_n(x, t)\, u(T^{(1)}) + \int_{T^{(1)}}^{T^{(2)}} k(s, x, t)\frac{du(s)}{ds}\, ds\,, \tag{23}$$

where $k(s, x, t) = T_n(x, t)r(s, x, t)$. Using for (23) the determination (6) of the conjugate operator and the scalar product (20), we obtain

$$\int_\Omega (K_{na}u)(x, t)\, w(x, t)\, dxdt = u(T^{(1)}) \int_\Omega \frac{\partial T_n(x, t)}{\partial x} w(x, t)\, dxdt + \int_{T^{(1)}}^{T^{(2)}} \frac{du(s)}{ds} \int_\Omega k(s, x, t)\, w(x, t)\, dxdtds =$$

$$= u(T^{(1)} \int_\Omega \frac{\partial T_n(x, t)}{\partial x} w(x, t)\, dxdt + \int_{T^{(1)}}^{T^{(2)}} \frac{du(s)}{ds} \frac{d}{ds}\left(\int_{T^{(1)}}^{s}\left(\int_\Omega k(\tau, x, t)\, w(x, t)\, dxdt\right) d\tau\right) ds =$$

$$= u(T^{(1)})\,(K_{na}^* w)(T^{(1)}) + \int_{T^{(1)}}^{T^{(2)}} \frac{du(s)}{ds} \frac{d}{ds}(K_{na}^* w)(s)\, ds\,.$$

This yields the representation (22) for the conjugate operator K_{n1}^*.

Theorem 3. For the scalar product (21), the $(K_{n1}^* w)(z)$ values of the conjugate operator K_{n1}^* may be computed according to the formula

$$(K_{n1}^* w)(z) = -\int_\Omega \frac{\partial T_n(x, t)}{\partial x} \frac{\partial w(x, t)}{\partial x}\, dxdt - \int_z^{T^{(2)}} \int_\Omega R(\tau, x, t) \frac{\partial T_n(x, t)}{\partial x} \frac{\partial w(x, t)}{\partial x}\, dxdtd\tau\,. \tag{24}$$

The proof of Theorem 3 is carried out analogously to the proof of Theorem 2.

Corollary 1. We introduce the notation

$$s_n = \int_\Omega \frac{\partial T_n(x, t)}{\partial x} \frac{\partial w_n(x, t)}{\partial x}\, dxdt\,, \quad l_{n1}(z) = \int_\Omega r(z, x, t) \frac{\partial T_n(x, t)}{\partial x} \frac{\partial w_n(x, t)}{\partial x}\, dxdt\,,$$

$$l_{n2}(z) = \int_\Omega R(z, x, t) \frac{\partial T_n(x, t)}{\partial x} \frac{\partial w_n(x, t)}{\partial x}\, dxdt\,.$$

Then, for the square norm of the gradient J'_{λ_n} in the space $W_2^1[T^{(1)}, T^{(2)}]$, we may write the formula ($i = 1$ corresponds to the scalar product (20) and $i = 2$ corresponds to the product (21))

$$\|J'_{\lambda_n}\|^2_{W_2^1} = \langle J'_{\lambda_n}, J'_{\lambda_n}\rangle = s_n^2 + \int_{T^{(1)}}^{T^{(2)}} l_{ni}^2(z)\, dz\,, \quad i \in \{1, 2\}\,.$$

Dependences (18), (22), and (24) determining the conjugate operator K_n^* in the spaces $L_2[T^{(1)}, T^{(2)}]$ and $W_2[T^{(1)}, T^{(2)}]$ may directly be used in numerical realization of the gradient methods for solution of inverse heat-conduction problems.

Conclusions. Within the framework of the theory of gradient methods, we have obtained algorithms of functional (without preliminary finite-dimensional approximation of $\lambda(T)$) identification of the nonlinear thermal-conductivity coefficient using new representations of the conjugate operators.

NOTATION

b, length of the segment, m; $c(T)$, heat-capacity coefficient, W·h/(m^3·^{o}C); J'_{λ_n}, gradient of the functional; G, open domain; t, running instant of time, h; t_f, final instant of time, h; T, temperature, ^{o}C; x, space coordinate, m; x_*, point of measurement of the temperature, m; X, Y, and $\tilde{X}$, Hilbert spaces; β_n, descent coefficient; $\lambda(T)$, thermal-conductivity coefficient, W/(m·^{o}C); Ω, domain of definition. Subscripts: f, final; n, iteration No. of the algorithm.

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